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ON THE FLOW OF A HEAVY FLUID IN A CHANNEL WITH A CURVILINEAR FLOOR

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O. M. KISELEV

(Kazan')

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Constructive proof is given of the single-valued solvability of the problem of flow of a heavy fluid with free surface in a channel with a curvilinear floor at fairly high Froude numbers and certain restrictions imposed on the floor shape. Nonconstructive proofs of the existence of solution with other restrictions on the floor shape were previously obtained in [1 — 3]. The proof of solvability in the case when the Froude number is reasonably close to but greater than unity appears in [4].

1. The stabilized flow of a perfect incompressible ponderable fluid bounded from above by free surface L and by a curvilinear floor S with horizontal asymptotes is considered in the plane $z = x + iy$ (Fig. 1). The coordinate origin is located on S with the y-axis directed vertically upward. At infinity upstream (to the left) the fluid flow is uniform and is defined by velocity V_0 and depth H of the stream.

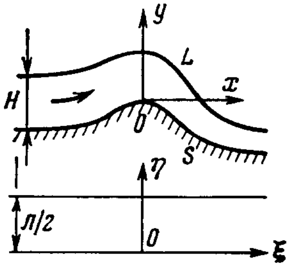

Fig. 1

Let l be the curvilinear abscissa of a point on S measured from the coordinate origin in the direction of flow, and β be the angle between the tangent to S in the direction of flow and the x-axis. We specify the shape of curve S by the equation

$$\beta = F(t), \quad t = l / H \quad (-\infty < t < \infty)$$

We assume function $F(t)$ to be twice differentiable and to satisfy for $-\infty < t < \infty$ the conditions

$$|F(t)| \leqslant B_0 e^{-b_0|t|}, \quad |F'(t)| \leqslant B_1 e^{-b_1|t|} |t|^{-1}$$
$$|F'(t)| \leqslant B_2, \quad |F''(t)| \leqslant B_3 |t|^{-1} \qquad (1.2)$$

where B_0, B_1, B_2, B_3, b_0 and b_1 are some positive constants.

Let the band $K = \{0 < \eta < \pi / 2\}$, conformally represent the flow region in the plane of the auxiliary variable $\zeta = \xi + i\eta$, with the straight lines $\eta = \pi / 2$ and $\eta = 0$ corresponding, respectively, to the free surface and to the solid boundary, and point $\zeta = 0$ to the coordinate origin of plane z. The complex flow potential w is defined by formula

$$w = 2V_0 H \pi^{-1} \zeta \qquad (1.3)$$

We introduce in the analysis the Joukowski function

$$f(\zeta) = \ln\left(V_0 \frac{dz}{dw}\right) = r + i\theta, \quad r = \ln\frac{V_0}{V} \tag{1.4}$$

where V is the modulus of velocity and θ is the angle of inclination of velocity to the x-axis. For defining the boundary values of function $f(\zeta)$ we use the notation

$$r_0 = \operatorname{Re} f(\xi), \quad \theta_0 = \operatorname{Im} f(\xi), \quad \theta_0' = d\theta_0 / d\xi$$

$$r_1 = \operatorname{Re} f(\xi + i\pi/2), \quad \theta_1 = \operatorname{Im} f(\xi + i\pi/2), \quad r'_1 = dr_1 / d\xi$$

We express the condition of constant pressure at the free surface in the form

$$r_1' e^{-3r_1} = \frac{g}{V_0^3}\frac{d\varphi}{d\xi}\sin\theta_1 \tag{1.5}$$

where g is the acceleration of gravity and φ is the velocity potential. In accordance with (1.3) and (1.4) we have

$$\frac{d\varphi}{d\xi} = \frac{2V_0 H}{\pi} \ \text{on } S \text{ and } L, \quad \frac{dt}{d\xi} = \frac{2}{\pi}e^{r_0} \ \text{on } S$$

Taking into account (1.1) and (1.5), we obtain for function $f(\zeta)$ the following nonlinear boundary value problem:

$$\theta_0' = \frac{2}{\pi}F'(t)e^{r_0}, \quad t = \frac{2}{\pi}\int_0^{\xi} e^{r_0}\,d\xi \tag{1.6}$$

$$r_1' = \alpha\sin\theta_1\left(1 - 3\alpha\int_{-\infty}^{\xi}\sin\theta_1\,d\xi\right)^{-1}, \quad \alpha = \frac{2gH}{\pi V_0^2}$$

$$\lim f(\zeta) = 0, \quad \xi \to -\infty$$

Function $f(\zeta)$, which is regular in K and continuous in the closed region K, is defined in terms of θ_0 and r_1 by formula [5]

$$f(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty}\{\theta_0(\tau)\operatorname{csch}(\tau - \zeta) + r_1(\tau)\operatorname{sch}(\tau - \zeta)\}\,d\tau \tag{1.7}$$

$$(\operatorname{csch} z = 1/\operatorname{sh} z, \ \operatorname{sch} z = 1/\operatorname{ch} z)$$

Assuming the boundedness and differentiability of functions θ_0 and r_1, from (1.7) by integrating by parts we obtain

$$r_0 = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\theta_0'(\tau)\ln\operatorname{cth}\frac{|\tau - \xi|}{2} + r_1(\tau)\operatorname{sch}(\tau - \xi)\right\}d\tau \tag{1.8}$$

$$\theta_1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\theta_0(\tau)\operatorname{sch}(\tau - \xi) + r_1'(\tau)\ln\operatorname{cth}\frac{|\tau - \xi|}{2}\right\}d\tau$$

Let $\varkappa(\xi)$ and $\lambda(\xi)$ be real functions determinate along the whole numerical axis. We introduce operators P_0, P_1, D_0, D_1, S_0, S_1 and K_0

$$P_0\varkappa = \int\limits_{-\infty}^{\xi} \alpha \sin\varkappa(\tau)\left(1 - 3\bar{\alpha}\int\limits_{-\infty}^{\tau}\sin\varkappa(\xi)\,d\xi\right)^{-1} d\tau \tag{1.9}$$

$$P_1\varkappa = \bar{\alpha}\sin\varkappa(\xi)\left(1 - 3\bar{\alpha}\int\limits_{-\infty}^{\xi}\sin\varkappa(\tau)\,d\tau\right)^{-1}$$

$$D_0\varkappa = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\varkappa(\tau)\,\mathrm{sch}\,(\tau - \xi)\,d\tau$$

$$D_1\varkappa = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\varkappa(\tau)\,\mathrm{lncth}\,\frac{|\tau - \xi|}{2}\,d\tau$$

$$S_0\varkappa = F(t), \quad S_1\varkappa = \frac{2}{\pi}e^{\varkappa(\xi)}F'(t), \quad t = \frac{2}{\pi}\int\limits_0^{\xi}e^{\varkappa(\tau)}\,d\tau$$

$$K_0(\varkappa, \lambda) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\{\varkappa(\tau)\,\mathrm{csch}\,(\tau - \zeta) + \lambda(\tau)\,\mathrm{sch}\,(\tau - \zeta)\}\,d\tau$$

In accordance with (1. 6)

$$r_1' = P_1\theta_1, \quad r_1 = P_0\theta_1, \quad \theta_0' = S_1r_0, \quad \theta_0 = S_0r_0$$

It follows from (1. 8) that functions r_0 and θ_1 satisfy the operator equations

$$r_0 = D_1S_1r_0 + D_0P_0\theta_1, \quad \theta_1 = D_0S_0r_0 + D_1P_1\theta_1 \tag{1.10}$$

Hence, if function $f(\zeta)$ is the solution of the boundary value problem (1. 6), the equalities (1. 10) are satisfied.

The converse statement is also valid. If the continuous and bounded real functions $\mu_0^*(\xi)$ and $\mu_1^*(\xi)$ are such that the limit

$$\lim_{a\to-\infty}\int\limits_a^0 \sin\mu_1^*\,d\xi = M, \quad |M| < \infty \tag{1.11}$$

exists and the relationships

$$\mu_0^* = D_1S_1\mu_0^* + D_0P_0\mu_1^*, \quad \mu_1^* = D_0S_0\mu_{01}^* + D_1P_1\mu_1^* \tag{1.12}$$

are satisfied, then function

$$h(\zeta) = K_0(S_0\mu_0^*, \; P_0\mu_1^*) \tag{1.13}$$

yields the solution of the boundary value problem (1. 6). In fact, function $h(\zeta)$ defined by formulas (1. 13) and (1. 9) is regular in K and continuous in the closed region $\overline{K}$. By comparing formulas (1. 7) and (1. 9) with allowance for (1. 8), we can conclude that

$$\begin{aligned}
&\mathrm{Im}\,h(\xi) = S_0\mu_0^*, \quad \mathrm{Re}\,h(\xi + i\pi/2) = P_0\mu_1^* \\
&\frac{d}{d\xi}\mathrm{Im}\,h(\xi) = S_1\mu_0^*, \quad \frac{d}{d\xi}\mathrm{Re}\,h\left(\xi + \frac{i\pi}{2}\right) = P_1\mu_1^* \\
&\mathrm{Re}\,h(\xi) = D_1S_1\mu_0^* + D_0P_0\mu_1^* \\
&\mathrm{Im}\,h(\xi + i\pi/2) = D_0S_0\mu_0^* + D_1P_1\mu_1^*
\end{aligned} \tag{1.14}$$

and in accordance with (1. 12)

$$\operatorname{Re} h(\xi) = \mu_0^*, \quad \operatorname{Im} h(\xi + i\pi/2) = \mu_1^*$$

Setting $f(\zeta) = h(\zeta)$, we conclude that conditions (1.6) are satisfied.

2. We assume that the real function $\varkappa(\xi)$ specified along the whole of the numerical axis belongs to space L_δ , if it is continuous and if for a fixed $\delta > 0$ there exists such $N > 0$ that

$$|\varkappa(\xi)| \leqslant N e^{-\delta|\xi|} \quad (< \infty < \xi < \infty) \tag{2.1}$$

We call the lowest value of N which satisfies inequality (2.1) the norm of function $\varkappa(\xi)$ in space L_δ and denote it by $\|\varkappa\|_\delta$. Space L_δ is a Banach space.

Let C be the space of real continuous and bounded functions. Introducing the Banach space

$$B = C \times L_\delta = \{\nu = (\mu_0, \mu_1) \colon \mu_0 \in C, \ \mu_1 \in L_\delta\}$$
$$(\|\nu\|_B = \|\mu_0\|_C + \|\mu_1\|_\delta)$$

and operator A defined in B by the equalities

$$A\nu = A_0\nu \times A_1\nu \tag{2.2}$$
$$A_0\nu = D_1 S_1 \mu_0 + D_0 P_0 \mu_1, \quad A_1\nu = D_0 S_0 \mu_0 + D_1 P_1 \mu_1$$

we write the system of Eqs. (1.12) in the form of the operator equation

$$\nu = A\nu \tag{2.3}$$

Let us investigate the properties of the introduced operators.

Lemma 1. For $0 < \delta < 1$ operators D_0 and D_1 transform space L_δ into itself, and

$$\|D_0\|_\delta \leqslant 4[\pi(1-\delta^2)]^{-1}, \quad \|D_1\|_\delta \leqslant \pi[2(1-\delta^2)]^{-1}$$

If $\varkappa(\xi) > L_\delta$ and $\delta < 1$, then

$$|D_1\varkappa| \leqslant \frac{\|\varkappa\|_\delta}{\pi} \lim \left\{ \int_{-R'}^{-\varepsilon'} e^{-\delta|\xi-\tau|} \ln\left|\operatorname{cth}\frac{\tau}{2}\right| d\tau + \right. \tag{2.4}$$
$$\left. \int_{\varepsilon}^{R} e^{-\delta|\xi-\tau|} \ln\left|\operatorname{cth}\frac{\tau}{2}\right| d\tau \right\} \qquad (0 < \varepsilon, \ \varepsilon' \to 0, \ R, R' \to \infty)$$

We substitute in formula (2.4) the series

$$\ln\left|\operatorname{cth}\frac{\tau}{2}\right| = 2\sum_{n=0}^{\infty}(2n+1)^{-1} e^{-(2n+1)|\tau|}$$

for $\ln|\operatorname{cth} {}^1/_2\,\tau|$. Changing in the last formula, first, the order of summation and integration and then, that of summation and passing to limit, we obtain

$$|D_1\varkappa| \leqslant \frac{4\|\varkappa\|_\delta}{\pi} \sum_{n=0}^{\infty} \frac{(2n+1)e^{-\delta|\xi|} - \delta e^{-(2n+1)|\xi|}}{(2n+1)[(2n+1)^2 - \delta^2]} \leqslant$$
$$\frac{4\|\varkappa\|_\delta}{\pi} e^{-\delta|\xi|} \sum_{n=0}^{\infty} [(2n+1)^2 - \delta^2]^{-1} < \frac{\pi\|\varkappa\|_\delta\, e^{-\delta|\xi|}}{2(1-\delta^2)}$$

Q.E.D. The validity of Lemma 1 for operator D_0 is proved with the use of the inequality $\operatorname{sch}\tau > 2e^{-|\tau|}$.

Lemma 2. Operators D_0 and D_1 transform space C into itself, and

$$\| D_0 \|_C = 1, \quad \| D_1 \|_C = \pi / 2$$

The proof of this statement is elementary.

We introduce the sets $C(R)$, $L_\delta(T)$ and $B(R, T)$

$$C(R) = \{\mu_0: \ \mu_0 \in C, \| \mu_0 \|_C \leqslant R\}$$

$$L_\delta(T) = \{\mu_1: \ \mu_1 \in L_\delta, \| \mu_1 \|_\delta \leqslant T\}$$

$$B(R, T) = \{\nu = (\mu_0, \mu_1): \mu_0 \in C(R), \ \mu_1 \in L_\delta(T)\}$$

Let $\nu = (\mu_0, \mu_1) \in B(R, T)$ when $0 < \delta < 1$. Using Lemmas 1 and 2, conditions (1.2), and the inequalities

$$|t| \geqslant 2\pi^{-1} e^{-R} |\xi|, \qquad \left| \int_{-\infty}^{\xi} \sin \mu_1 d\xi \right| \leqslant 2T\delta^{-1}$$

we can show that $A\nu \in B(R, T)$ only if the relationships

$$6\alpha T\delta^{-1} < 1, \quad 2\pi^{-1} b_0 e^{-R} \geqslant \delta \tag{2.5}$$

$$\frac{4B_0}{\pi(1-\delta^2)} + \frac{\pi}{2(1-\delta^2)} \frac{\alpha T}{1 - 6\alpha T\delta^{-1}} \leqslant T$$

$$B_2 e^R + 2\alpha T\delta^{-1} (1 - 6\alpha T\delta^{-1}) \leqslant R$$

are satisfied.

Let $\nu' = (\mu_0', \mu_1') \in B(R, T)$, $\nu'' = (\mu_0'', \mu_1'') \in B(R, T)$ and the first of inequalities (2.5) be satisfied. Then

$$\| P_1\mu_1' - P_1\mu_1'' \|_\delta \leqslant \frac{\alpha(1 + 12\alpha T\delta^{-1})}{(1 - 6\alpha T\delta^{-1})^2} \| \mu_1' - \mu_1'' \|_\delta \tag{2.6}$$

$$\| P_0\mu_1' - P_0\mu_1'' \|_C \leqslant \frac{2}{\delta} \frac{\alpha(1 + 12\alpha T\delta^{-1})}{(1 - 6\alpha T\delta^{-1})^2} \| \mu_1' - \mu_1'' \|_\delta$$

$$| S_0\mu_0' - S_0\mu_0'' | \leqslant B_1 e^{2R} \exp\left(-\frac{2}{\pi} b_1 e^{-R} |\xi|\right) \| \mu_0' - \mu_0'' \|_C$$

$$\| S_1\mu_0' - S_1\mu_0'' \| \leqslant \frac{2}{\pi} e^R (B_2 + B_3 e^{2R}) \| \mu_0' - \mu_0'' \|_C$$

Under condition

$$2\pi^{-1} b_1 e^{-R} > \delta \tag{2.7}$$

we obtain

$$\| A\nu' - A\nu'' \|_B = \| A_0\nu' - A_0\nu'' \|_C + \| A_1\nu' - A_1\nu'' \|_\delta \leqslant \gamma (\| \mu_0' - \mu_0'' \|_C + \| \mu_1' - \mu_1'' \|_\delta)$$

$$\gamma < \max\left\{[4B_1[\pi(1-\delta^2)]^{-1} + B_2 + B_3] e^{3R} \left[\frac{\pi}{2(1-\delta^2)} + \frac{2}{\delta}\right] \frac{\alpha(1 + 12\alpha T\delta^{-1})}{(1 - 6\alpha T\delta^{-1})^2}\right\}$$

Thus operator A transforms the set $B(R, T)$ into itself and is on that set a contraction operator if the inequalities (2.5), (2.7) and

$$\left[\frac{\pi}{2(1-\delta^2)}+\frac{2}{\delta}\right]\frac{\alpha(1+12\alpha T\delta^{-1})}{(1-6\alpha T\delta^{-1})^2}\leqslant 1 \tag{2.8}$$

$$\{4B_1[\pi(1-\delta^2)]^{-1}+B_2+B_3\}e^{3R}\leqslant 1 \tag{2.9}$$

are satisfied.

3. Let us write the first and third of inequalities (2.5) in the form

$$\omega<1,\quad \beta_1+\gamma_1\omega(1-\omega)^{-1}\leqslant\omega \tag{3.1}$$

$$\omega=6\ddot{\alpha}T\delta^{-1},\quad B_1=\frac{24B_0\alpha\delta^{-1}}{\pi(1-\delta^2)},\quad \gamma_1=\frac{\pi\alpha}{2(1-\delta^2)}$$

The analysis of inequalities (3.1) shows that they are satisfied when conditions

$$\beta_1+\gamma_1\leqslant{}^1/_2,\quad \omega=\omega_0={}^1/_2(\gamma_1+3\beta_1)$$

are fulfilled. These conditions are equivalent to the following:

$$\alpha\leqslant P={}^1/_2\{24B_0\delta^{-1}[\pi(1-\delta^2)]^{-1}+\pi[2(1-\delta^2)]^{-1}\}^{-1}$$
$$T=T_0=6B_0[\pi(1-\delta^2)]^{-1}+\pi\delta[24(1-\delta^{-2})]^{-1}$$

Setting $T=T_0$ we write (2.8) as

$$\omega_0(1+2\omega_0)(1-\omega_0)^{-2}\leqslant\varkappa_1,\quad \omega_0=6\alpha T_0\delta^{-1} \tag{3.2}$$

$$\varkappa_1=6T_0\delta^{-1}\{\pi[2(1-\delta^2)]^{-1}+2\delta^{-1}\}^{-1}$$

Taking into account the requirement for $\omega_0<1$, from (3.2) we obtain

$$\ddot{\alpha}\leqslant Q=\begin{cases}\delta(1+2\varkappa_1-\sqrt{1+12\varkappa_1})[6T_0(2\varkappa_1-4)]^{-1} & (\varkappa_1\neq 2)\\ \delta(15T_0)^{-1} & (\varkappa_1=2)\end{cases}$$

If the second and fourth of inequalities (2.5) and the inequalities (2.7) and (2.9) are to be satisfied, it is necessary to impose certain restrictions on the constants that define function $F(t)$. We assume that

$$B_2\leqslant(e-1)e^{-2} \tag{3.3}$$

$$\min\left\{\frac{1}{3}(1-eB_2)\ln\left[\frac{4B_1}{\pi(1-\delta^2)}+B_2+B_3\right]^{-1}-B_2\right. \tag{3.4}$$
$$\left.-(1-eB_2)\ln(2b_k/\delta\pi)-B_2\ (k=0,1)\right\}=C_0>0$$

Setting $T=T_0$ we write (2.5) in the form

$$B_2e^R+D\leqslant R,\quad D=2\alpha T_0\delta^{-1}(1-6\alpha T_0\delta^{-1})^{-1} \tag{3.5}$$

Taking into consideration (3.3) it is possible to show that the inequality is satisfied when

$$D\leqslant e^{-1},\quad R=\rho(\alpha)=(B_2+D)(1-eB_2)^{-1} \tag{3.6}$$

In accordance with (3.4) the second of inequalities (2.5) and the inequalities (2.7) and (2.9) are satisfied for $R=\rho(\alpha)$ if

$$D\leqslant C_0 \tag{3.7}$$

Conditions (3.6) and (3.7) are equivalent to the following:

$$\alpha \leqslant U = \delta C_1 \, [2T_0 (1 + 3C_1)]^{-1}, \quad C_1 = \min (C_0, e^{-1})$$

Thus, when constants B_1, B_2, B_3, b_0 and b_1 satisfy relationships (3.3) and (3.4), then with $T = T_0$, $R = \rho(\alpha)$ and $\alpha \leqslant \alpha_0 = \min (P, Q, U)$ the inequalities (2.5) and (2.7) — (2.9) are fulfilled.

The following theorem is formulated with the use of the principle of compressed mapping.

Theorem 1. If conditions (1.2), (3.3), (3.4) and $0 < \delta < 1$, then for $\alpha = \alpha_1 < \alpha_0$ there exists in space $B' = B(\rho(\alpha_1), T_0)$ the solution $\nu^* = (\mu_0^*, \mu_1^*)$ of Eq. (2.3) which is unique in the space $B'' = B(\rho(\alpha_0), T_0)$ $(B' \subset B'')$. That solution can be found as the limit of sequence

$$\nu^{(n)} = A\nu^{(n-1)} \quad (n = 1, 2, \ldots) \tag{3.8}$$

for any initial approximation $\nu^{(0)} \in B'$. The estimate of the n-th approximation error is given by formula

$$\| \nu^* - \nu^{(n)} \|_B \leqslant \frac{\gamma_1^{\,n}}{1 - \gamma_1} \| \nu^{(0)} - A\nu^{(0)} \|_B \tag{3.9}$$

$$\gamma_1 = \max \{ [4B_1 (\pi (1 - \delta^2))^{-1} + B_2 + B_3] e^{3\rho(\alpha_1)}$$

$$\left[\frac{\pi}{2(1 - \delta^2)} + \frac{2}{\delta} \right] \frac{\alpha_1 (1 + 12\alpha_1 T_0 \delta^{-1})}{(1 - 6\alpha_1 T_0 \delta^{-1})^2} \}$$

4. Let $\nu^* = (\mu_0^*, \mu_1^*)$ be the solution of Eq. (2.3) and $\nu^* \in B$. Then, as shown above, function $f(\zeta)$ which yields the solution of the boundary value problem (1.6) is defined by formula

$$f(\zeta) = K_0 (S_0\mu_0^*, P_0\mu_1^*)$$

with

$$r_0 = \mu_0^*, \quad \theta_1 = \mu_1^*, \quad \theta_0 = S_0\mu_0^*, \quad r_1 = P_0\mu_1^* \tag{4.1}$$

Let $\nu^{(0)} \in B'$, $\nu^{(n-1)} = (\mu_0^{(n-1)}$ and $\mu_1^{(n-1)})$, $(n = 1, 2, \ldots)$. We introduce the notation

$$f^{(n)}(\zeta) = K_0 (S_0\mu_0^{(n-1)}, P_0\mu_1^{(n-1)}) \quad (n = 1, 2, \ldots) \tag{4.2}$$

$$\lambda_0^{(n)} = \operatorname{Im} f^{(n)}(\xi), \quad \lambda_1^{(n)} = \operatorname{Re} f^{(n)}(\xi + i\pi / 2)$$

Function $f^{(n)}(\zeta)$ is regular in K and continuous in $\bar{K}$. In accordance with (1.13) and (1.14)

$$\lambda_0^{(n)} = S_0\mu_0^{(n-1)}, \quad \lambda_1^{(n)} = P_0\mu_1^{(n-1)} \tag{4.3}$$

$$\operatorname{Re} f^{(n)}(\xi) = D_1 S_1 \mu_0^{(n-1)} + D_0 P_0 \mu_1^{(n-1)}$$

$$\operatorname{Im} f^{(n)}(\xi + i\pi / 2) = D_0 S_0 \mu_0^{(n-1)} + D_1 P_1 \mu_1^{(n-1)}$$

Taking into consideration (2.2) and (3.8) we obtain

$$\operatorname{Re} f^{(n)}(\xi) = \mu_0^{(n)}, \quad \operatorname{Im} f^{(n)}(\xi + i\pi/2) = \mu_1^{(n)}$$

Allowing for (4.1) and (4.3) we have

$$\max | f(\zeta) - f^{(n)}(\zeta) | \leqslant \max \{ \| r_0 - \mu_0^{(n)} \|_C + \| \theta_0 - \lambda_0^{(n)} \|_C, \quad \| r_1 -$$
$$- \lambda_1^{(n)} \|_C + \| \theta_1 - \mu_1^{(n)} \|_C \} \leqslant \max \{ \| \mu_0^* - \mu_0^{(n)} \|_C +$$
$$+ \| S_0\mu_0^* - S_0\mu_0^{(n-1)} \|_\delta$$
$$\| P_0\mu_1^* - P_0\mu_1^{(n-1)} \|_C + \| \mu_1^* - \mu_1^{(n)} \|_\delta \}$$

When the conditions of Theorem 1 are satisfied with allowance for the inequalities (2.6) and (3. 9), we obtain

$$\max_{\zeta\in\bar{K}} | f(\zeta) - f^{(n)}(\zeta) | \leqslant \frac{2\gamma_1{}^n}{1-\gamma_1} \| \nu^{(0)} - A\nu^{(0)} \|_B \qquad (4.4)$$

We denote by $M\ (R,\ T)$ the class of functions $f\ (\zeta)$ that are regular in K and continuous in $\overline{K}$, and such that

$$| \operatorname{Re} f\ (\xi) | \leqslant R, \quad | \operatorname{Im} f\ (\xi + i\pi\ /\ 2) | \leqslant T$$

Using the obtained results it is possible to formulate the following basic theorem.

Theorem 2. When conditions (1. 2),(3. 3),(3. 4) and $0 < \delta < 1$ are satisfied, then for $\alpha = \alpha_1 < \alpha_0$ there exists in the class $M\ (\rho\ (\alpha_1),\ T_0)$ a solution of the boundary value problem (1. 6) which in class $M\ (\rho\ (\alpha_0),\ T_0)$ is unique. That solution can be obtained as the limit of sequence (4. 2) for any $\nu^{(0)} \in B'$. The estimate of the n-th approximation error is given by formula (4. 4).

If the Joukowski function $f\ (\zeta)$ is known, function $z\ (\zeta)$ which maps band K on the region of flow is determined by formula

$$z\,(\zeta) = \frac{2H}{\pi} \int_0^{\zeta} e^{f(\zeta)} d\zeta \qquad (4.5)\ .$$

By analyzing successive approximations it is possible to prove by mathematical induction the following two theorems.

Theorem 3. When conditions (1. 2),(3. 3),(3. 4), $0 < \delta < 1$, $\alpha = \alpha_1 < \alpha_0$ and $F\ (t) = -F\ (-\ t)$, then function $f\ (\zeta)$ which belongs to class $M\ (\rho\ (\alpha_1),\ T_0)$ and yields the solution of the boundary value problem (1. 6) us such that

$$f\,(-\bar{\zeta}) = \overline{f\,(\zeta)}, \qquad \zeta \in \overline{K}$$

In other words, when the channel floor is symmetric about the y-axis, the fluid flow that corresponds to the obtained solution is also symmetric about that axis.

Theorem 4. When conditions (1.2),(3.3),(3.4). $0<\delta<1$, $\alpha = \alpha_1 < \alpha_0$, $T_0 < \pi$ and $F\ (t) \leqslant 0\ \ (F\ (t) \geqslant 0)$, then function $f\ (\zeta)$ which belongs to class $M\ (\rho\ (\alpha_1),\ T_0)$, and provides the solution of the boundary value problem (1. 6) is such that $r_1{}'$ and $\theta_1 \leqslant 0$ ($r_1{}'$ and $\theta_1 \geqslant 0$).

In other words, when the channel floor drops (rises) in the flow direction, the free surface behaves similarly. The stream depth at infinity to the right (of the y-axis) is smaller (greater) than that of the stream depth H at infinity to the left (of that axis).

Conditions (1. 2), (3. 3) and (3. 4) appearing in Theorems 1 — 4 differ as to the restrictions imposed on the floor shape from those used by the authors of papers [1 — 3]. The flow in a channel with a monotonically dropping floor ($F\ (t) \leqslant 0$) was investigated in [1], while papers [2, 3] dealt with the flow past an obstruction on a horizontal floor ($F\ (t) = 0$ with $| t | \geqslant t_0 > 0$). The three authors stipulated moreover the fulfillment of the inequality $| F\ (t) | < \pi\ /\ 2$. Conditions (1. 2), (3. 3) and (3. 4) do not contain any of these restrictions.

Let us prove that $| F\ (t) |$ can exceed $\pi\ /\ 2$. Let, for example,

$$F\,(t) = c \int_{-\infty}^{t} \frac{\tau d\tau}{(1+\tau^2) \operatorname{ch} a\tau}, \quad c, a > 0$$

It can be readily ascertained that conditions (1. 2) are satisfied when $b_0 = b_1 = a$, $B_0 = 2c\ /\ a$, $B_1 = 2c$, $B_2 = c\ /\ 2$ and $B_3 = c\ (1 + a)$. The lower estimate of max $|F(t)|$ is

$$\max|F(t)| = c\int_{\infty}^{0}\frac{\tau d\tau}{(1+\tau^2)\operatorname{ch} a\tau} > \frac{c}{2}\int_{1}^{\infty}\frac{d\tau}{\tau\operatorname{ch} a\tau} >$$

$$\frac{c}{2}\left\{\frac{1}{2}\int_{1}^{2}e^{-a\tau}d\tau + \frac{1}{3}\int_{2}^{3}e^{-a\tau}d\tau + \frac{1}{4}\int_{3}^{4}e^{-a\tau}d\tau + \ldots\right\} =$$

$$\frac{1}{2}ca^{-1}(e^{a}-1)[-\ln(1-e^{-a})-e^{-a}]$$

The positive constants c, a and δ can be, evidently, chosen so that with conditions (3.3) and (3.4) satisfied $\max|F(t)|$ exceeds any a priori specified number. Note that $\alpha_0 \to 0$ when a and $\delta \to 0$, i.e. when $\max|F(t)| \to \infty$.

5. For computation purposes it is convenient to pass in Eqs. (1.10) from variables ξ and τ to σ and u defined by

$$\xi = \ln\operatorname{ctg}{}^1/_2\,\sigma, \quad \tau = \ln\operatorname{ctg}{}^1/_2\,u, \quad 0 \leqslant \sigma,\ u \leqslant \pi$$

The functions and operators obtained by such substitution will be denoted by a dot superscript. We have

$$\varkappa(\xi(\sigma)) = \varkappa^{\cdot}(\sigma), \quad \varkappa(\tau(u)) = \varkappa^{\cdot}(u)$$

$$P_0\varkappa = P_0^{\cdot}\varkappa^{\cdot} = -\int_{\pi}^{\sigma}\frac{\alpha\sin\varkappa^{\cdot}(u)}{\sin u}\left(1+3\alpha\int_{\pi}^{u}\frac{\sin\varkappa^{\cdot}(\sigma)}{\sin\sigma}d\sigma\right)^{-1}du$$

$$P_1\varkappa = P_1^{\cdot}\varkappa^{\cdot} = \alpha\sin\varkappa^{\cdot}(\sigma)\left(1+3\alpha\int_{\pi}^{\sigma}\frac{\sin\varkappa^{\cdot}(u)}{\sin u}du\right)^{-1}$$

$$S_0\varkappa = S_0^{\cdot}\varkappa^{\cdot} = F(t), \quad t = -\frac{2}{\pi}\int_{\pi/2}^{\sigma}e^{\varkappa^{\cdot}(u)}\frac{du}{\sin u}$$

$$S_1\varkappa = S_1^{\cdot}\varkappa^{\cdot} = \frac{2}{\pi}e^{\varkappa^{\cdot}(\sigma)}F'(t)$$

$$D_0\varkappa = D_0^{\cdot}\varkappa^{\cdot} = \frac{\sin\sigma}{\pi}\int_{0}^{\pi}\frac{\varkappa^{\cdot}(u)\,du}{1-\cos u\cos\sigma}$$

$$D_1\varkappa = D^{\cdot}\varkappa^{\cdot} = \frac{1}{\pi}\int_{0}^{\pi}\frac{\varkappa^{\cdot}(u)}{\sin u}\ln\left|\frac{\operatorname{tg}{}^1/_2u + \operatorname{tg}{}^1/_2\sigma}{\operatorname{tg}{}^1/_2u - \operatorname{tg}{}^1/_2\sigma}\right|du$$

Equations (1.10) now assume the form

$$r^{\cdot}_0 = D_1^{\cdot}S_1^{\cdot}r_0^{\cdot} + D_0^{\cdot}P_0^{\cdot}\theta_1^{\cdot}, \quad \theta_1^{\cdot} = D_0^{\cdot}S_0^{\cdot}r_0^{\cdot} + D_1^{\cdot}P_1^{\cdot}\theta_1^{\cdot} \qquad (5.1)$$

Let function $\varkappa^{\cdot}(\sigma)$ specified in the interval $[0, \pi]$ be representable in the form of a Fourier series in cosines

$$\varkappa^{\cdot}(\sigma) = \sum_{k=0}^{\infty}B_k\cos k\sigma \qquad (5.2)$$

Taking into account that

$$\ln\left|\frac{\operatorname{tg}{}^1/_2\sigma+\operatorname{tg}{}^1/_2u}{\operatorname{tg}{}^1/_2\sigma-\operatorname{tg}{}^1/_2u}\right|=2\sum_{n=1}^{\infty}\frac{\sin n\sigma\sin nu}{n}$$

and using formulas 3.612 and 3.613 from [6], we obtain

$$D_0^{\cdot}\varkappa^{\cdot}=\sum_{k=0}^{\infty}B_k\left(\frac{1-\sin\sigma}{\cos\sigma}\right)^k \tag{5.3}$$

$$D_1^{\cdot}\varkappa^{\cdot}=2\sum_{m=1}^{\infty}\left\{\frac{\sin(2m-1)\sigma}{2m-1}\sum_{k=0}^{m-1}B_{2k}+\frac{\sin 2m\sigma}{2m}\sum_{k=1}^{m}B_{2k-1}\right\}$$

The solution of Eq. (2.3) by the scheme (3.8) is equivalent to the determination of functions $r_0^{\cdot}(\sigma)$ and $\theta_1^{\cdot}(\sigma)$ by using Eqs. (5.1) and the scheme

$$r_0^{\cdot(n+1)}=D_1^{\cdot}S_1^{\cdot}r_0^{\cdot(n)},+D_0^{\cdot}P_0^{\cdot}\theta_1^{\cdot(n)},\quad \theta_1^{\cdot(n+1)}=D_0^{\cdot}S_0^{\cdot}r_0^{\cdot(n)}+D_1^{\cdot}P_1^{\cdot}\theta_1^{\cdot(n)} \tag{5.4}$$

If the conditions of Theorem 1 are satisfied and $r_0^{\cdot(0)}\equiv\theta_1^{\cdot(0)}\equiv 0$, then for any n functions $r_0^{\cdot(n)}$ and $\theta_1^{\cdot(n)}$ are continuous, and

$$r_0^{\cdot(n)}(\pi)=0,\quad |\theta_1^{\cdot(n)}(\sigma)|\leqslant M(\sin\sigma)^{\delta}$$

where M is some constant. Assuming that functions $r_0^{\cdot(n)}$ and $\theta_1^{\cdot(n)}$ have been determined, we expand $S_0^{\cdot}r_{(0)}^{\cdot(n)}$, $S_1^{\cdot}r_0^{\cdot(n)}$, $P_0^{\cdot}\theta_1^{\cdot(n)}$ and $P_1^{\cdot}\theta_1^{\cdot(n)}$ into Fourier series in cosines

$$S_0^{\cdot}r_0^{\cdot(n)}=\sum_{k=0}^{\infty}a_k^{(n)}\cos k\sigma,\quad S_1^{\cdot}r_0^{\cdot(n)}=\sum_{k=0}^{\infty}b_k^{(n)}\cos k\sigma$$

$$P_0^{\cdot}\theta_1^{\cdot(n)}=\sum_{k=0}^{\infty}c_k^{(n)}\cos k\sigma,\quad P_1^{\cdot}\theta_1^{\cdot(n)}=\sum_{k=0}^{\infty}d_k^{(n)}\cos k\sigma$$

In accordance with (5.2)—(5.4) functions $r_0^{\cdot(n+1)}$ and $\theta_1^{\cdot(n+1)}$ are defined by formulas

$$\begin{Bmatrix}r_0^{\cdot(n+1)}\\ \theta_1^{\cdot(n+1)}\end{Bmatrix}=\sum_{k=0}^{\infty}\begin{Bmatrix}c_k^{(n)}\\ a_k^{(n)}\end{Bmatrix}\left(\frac{1-\sin\sigma}{\cos\sigma}\right)^k+$$

$$2\sum_{m=1}^{\infty}\left[\frac{\sin(2m-1)\sigma}{2m-1}\sum_{k=0}^{m-1}\begin{Bmatrix}b_{2k}^{(n)}\\ d_{2k}^{(n)}\end{Bmatrix}+\frac{\sin 2m\sigma}{2m}\sum_{k=1}^{m}\begin{Bmatrix}b_{2k-1}^{(n)}\\ d_{2k-1}^{(n)}\end{Bmatrix}\right]$$

Taking into consideration the asymptotic properties of the mapping function $z(\zeta)$ when $\xi\to-\infty$, the shape of the channel floor $(x_0(\sigma), y_0(\sigma))$ and the free surface $(x_1(\sigma), y_1(\sigma))$ can be determined in accordance with (4.5) by formulas

$$\frac{x_0}{H}=-\frac{2}{\pi}\int_{\pi/2}^{\sigma}e^{r_0^{\cdot}(u)}\frac{\cos\theta_0^{\cdot}(u)}{\sin u}\,du \tag{5.5}$$

$$\frac{y_0}{H}=-\frac{2}{\pi}\int_{\pi/2}^{\sigma}e^{r_0^{\cdot}(u)}\frac{\sin\theta_0^{\cdot}(u)}{\sin u}\,du$$

$$\frac{x_1}{H}=-\frac{2}{\pi}\int_{\pi-\varepsilon}^{\sigma}e^{r_1^{\cdot}(u)}\frac{\cos\theta_1^{\cdot}(u)}{\sin u}\,du$$

$$\frac{y_1}{H} = -\frac{2}{\pi} \int_{\pi-\varepsilon}^{\sigma} e^{r_1^{\cdot}(u)} \frac{\sin \theta_1^{\cdot}(u)}{\sin u} du + 1 + \frac{y_\varepsilon}{H}$$

$$\frac{x_\varepsilon}{H} = -\frac{2}{\pi} \int_{\pi/2}^{\pi-\varepsilon} e^{r_0^{\cdot}(u)} \frac{\cos \theta_0^{\cdot}(u)}{\sin u} du, \quad \theta_0^{\cdot} = S_0^{\cdot} r_0^{\cdot}$$

$$\frac{y_\varepsilon}{H} = -\frac{2}{\pi} \int_{\pi/2}^{\pi-\varepsilon} e^{r_0^{\cdot}(u)} \frac{\sin \theta_0^{\cdot}(u)}{\sin u} du, \quad r_1^{\cdot} = P_0^{\cdot} \theta_1^{\cdot}$$

where ε is a reasonably small positive quantity.

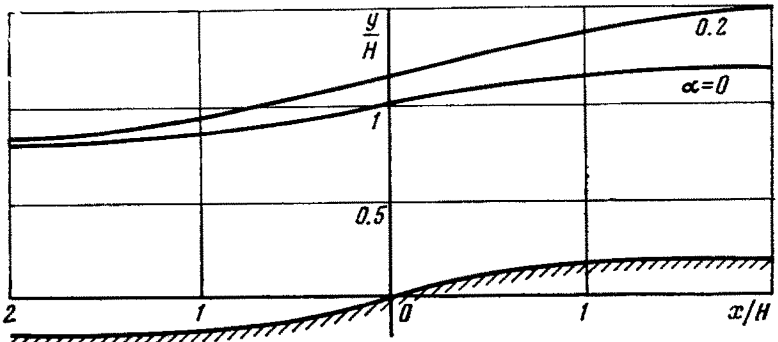

Fig. 2

The described method was used for computing the flow in a channel for

$$F(t) = 0.3 \,(\operatorname{ch} {}^1/_2\, \pi t)^{-2}$$

The flow boundaries are shown in Fig. 2 for $\alpha = 0$ (imponderable fluid) and $\alpha = 0.2$. The shape of the channel floor computed by formulas (5.5) for these values were virtually the same as the shape determined by formulas

$$\frac{x_0}{H} = \int_0^t \cos F(t)\, dt, \quad \frac{y_0}{H} = \int_0^t \sin F(t)\, dt$$

which indicates a fairly high accuracy of obtained here results.

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